Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(02)00029-1
J. Appl. Maths Mechs, Vol. 66, No. 2, pp. 239–243, 2002

0021-8928/02/$—see front matter

THE UNSTEADY MOTION OF A VISCOUS FLUID BETWEEN ROTATING PARALLEL WALLS†

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(*Received* 12 *November* 2001)

The motions of a viscous incompressible fluid, rotating initially as a rigid body with constant angular velocity together with parallel walls which bound it, acted upon by suddenly starting longitudinal oscillations of one of the walls, are investigated. The walls make an arbitrary angle with the axis of rotation. In general, the solution is obtained in the form of the sum of an infinite series and is represented by an integral containing an elliptic function. A number of special cass of the motion of the wall is considered. The results obtained are used to investigate certain structures of the boundary layers on the walls.

This paper generalizes previous results [1–4]. It is shown that, as the fixed wall is moved to infinity, the solution becomes identical with the result obtained in [4], and when there is no rotation and the fixed wall is moved to infinity, the solution changes into the well-known solution of the problem of the unsteady motion of a fluid bounded by a moving plane wall [1].

1. EXACT SOLUTIONS OF THE NAVIER–STOKES EQUATIONS

Consider the motion of a viscous incompressible fluid, which fills a gap of width l between parallel planes Q_0 and Q_1, which rotate with angular velocity $\boldsymbol{\omega}_0 = \text{const}$, where the vector $\boldsymbol{\omega}_0$ makes a constant angle $\beta(0 < \beta \leqslant \pi/2)$ with these planes. The special case when $\beta = \pi/2$ was investigated earlier in [2]. The plane Q_0 at the instant of time $t = 0$ begins to move in a longitudinal direction with velocity $\mathbf{u}(t)$. We will associated a Cartesian system of coordinates $Oxyz$ with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ with the rotating plane so that the Oxz plane coincides with the Q_0 plane, while the y axis is directed along the normal to it inside the fluid.

The equations of motion of the fluid in the $Oxyz$ system, rotating with angular velocity $\boldsymbol{\omega}_0$, and also the boundary and initial conditions have the form

$$\boldsymbol{\omega}_0(\boldsymbol{\omega}_0 \times \mathbf{r}) + 2\boldsymbol{\omega}_0 \times \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \nabla U + \nu\Delta\mathbf{v}, \quad \nabla\mathbf{v} = 0$$
$$\mathbf{v}(\mathbf{r},\ t) = \mathbf{u}(t), \quad \mathbf{r} \in Q_0,\ t > 0; \quad \mathbf{v}(\mathbf{r},\ t) = 0, \quad \mathbf{r} \in Q_1; \quad \mathbf{v}(\mathbf{r},\ 0) = 0 \tag{1.1}$$

Here $\mathbf{r}$ is the radius vector relative to the pole O, v is the fluid velocity, P is the pressure, ρ is the density, ν is the kinematic viscosity of the fluid and U is the potential of the external mass forces. The solution of system of equations (1.1) will be sought in the form

$$P = \rho(\boldsymbol{\omega}_0 \times \mathbf{r})^2/2 + \rho U + \rho q(y,\ t), \quad \mathbf{v} = \upsilon_x(y,\ t)\mathbf{e}_x + \upsilon_z(y,\ t)\mathbf{e}_z$$

Then system (1.1) can be split into two subsystems

$$\frac{\partial \mathbf{v}}{\partial t} + 2\Omega(\mathbf{e}_y \times \mathbf{v}) = \nu\frac{\partial^2 \mathbf{v}}{\partial y^2}, \quad \Omega = \boldsymbol{\omega}_0 \cdot \mathbf{e}_y; \quad \frac{\partial q}{\partial y} = 2\mathbf{v}(\boldsymbol{\omega}_0 \times \mathbf{e}_y) \tag{1.2}$$
$$\mathbf{v}(0,\ t) = \mathbf{u}(t), \quad \mathbf{v}(l,\ t) = 0, \quad \mathbf{v}(y,\ 0) = 0$$

Here the velocity field is found from the first equation of (1.2) while the pressure field is found from the velocity field obtained from the second equation of (1.2).

†*Prikl. Mat. Mekh.* Vol. 66, No. 2, pp. 251–255, 2002.

We will seek a solution of system (1.2) in the form

$$\mathbf{v} = \mathbf{W}\sin 2\Omega t - (\mathbf{W}\times\mathbf{e}_y)\cos 2\Omega t, \quad \mathbf{W} = W_x(y,\ t)\mathbf{e}_x + W_z(y,\ t)\mathbf{e}_z \tag{1.3}$$

where $\mathbf{W}(x, y)$ is a new unknown function. Substituting (1.3) into (1.2) we obtain the following problem for determining W

$$\mathbf{W}_t = \nu\mathbf{W}_{yy}, \quad 0 \leqslant y \leqslant l \tag{1.4}$$
$$\mathbf{W}(0,\ t) = \mathbf{v}(t)\sin 2\Omega t + (\mathbf{v}(t)\times\mathbf{e}_y)\cos 2\Omega t, \quad \mathbf{W}(l,\ t) = 0; \quad \mathbf{W}(y,\ 0) = 0$$

Equation (1.4) has a system of eigenfunctions

$$\sin\frac{\pi n}{l}y, n = 1, 2, \ldots$$

We will introduce an auxiliary function $z(y, t)$ by the equation

$$\mathbf{W} = (1 - y/l)\mathbf{W}(0,\ t) + \mathbf{z}(y,\ t) \tag{1.5}$$

Substituting (1.5) into (1.4) we obtain an inhomogeneous equation with homogeneous initial and boundary conditions

$$\frac{\partial\mathbf{z}}{\partial t} = \nu\frac{\partial^2\mathbf{z}}{\partial y^2} - \left(1 - \frac{y}{l}\right)\frac{\partial\mathbf{W}(0,\ t)}{\partial t} \tag{1.6}$$
$$\mathbf{z}(0,\ t) = 0, \quad \mathbf{z}(l,\ t) = 0, \quad \mathbf{z}(y,\ 0) = 0.$$

We will seek a solution of Eqs (1.6) in the form of a series in eigenfunctions of the homogeneous equation (1.4) (everywhere henceforth, unless otherwise stated, summation is carried out over n from 1 to ∞)

$$\mathbf{z} = \sum\mathbf{z}_n(t)\sin(\pi ny/l) \tag{1.7}$$

Substituting series (1.7) into Eq. (1.6), we obtain a first-order equation, whence we find

$$\mathbf{z}_n(t) = -\frac{2}{\pi n}\int_0^t \dot{\mathbf{W}}(0,\ \tau)\exp[-\nu\lambda_n^2(t-\tau)]d\tau, \text{ where } \lambda_n = \frac{\pi n}{l} \tag{1.8}$$

(the dot denotes a derivative with respect to t). Integrating Eq. (1.8) on the right-hand side by parts and substituting into Eq. (1.5), taking representation (1.7) into account, we obtain the following solution of Eq. (1.4)

$$\mathbf{W} = 2\pi\nu l^{-2}\sum n\sin(\lambda_n y)\int_0^t \mathbf{W}(0,\ \tau)\exp[-\nu\lambda_n^2(t-\tau)]d\tau \tag{1.9}$$

Using formula (1.9), from (1.3) we find the velocity field

$$\mathbf{v} = 2\pi\nu l^{-2}\sum n\sin(\lambda_n y)\mathbf{B}_n(t), \quad \mathbf{B}_n(t) = \int_0^t \mathbf{T}(\tau,\ t-\tau)\exp[-\nu\lambda_n^2(t-\tau)]d\tau \tag{1.10}$$
$$\mathbf{T}(\tau,\ t-\tau) = \mathbf{u}(\tau)\cos 2\Omega(t-\tau) + (\mathbf{u}(\tau)\times\mathbf{e}_y)\sin 2\Omega(t-\tau)$$

Solving the second of equations (1.2), into the right-hand side of which we have substituted expression (1.10), we obtain the pressure field inside the gap. The vectors of the shear stresses acting on the boundary of the gap from the fluid side, are given by the expressions

$$\mathbf{F}_0 = 2\pi^2\rho\nu^2 l^{-3}\sum n^2\mathbf{B}_n(t), \quad \mathbf{F}_l = 2\pi^2\rho\nu^2 l^{-3}\sum(-1)^n n^2\mathbf{B}_n(t) \tag{1.11}$$

Relations (1.10) and (1.11) completely solve the problem.

For our further analysis it is convenient to represent the velocity field and the vectors of the shear stresses in the complex form

$$\hat{v} = v_z + iv_x, \hat{u} = u_z + iu_x, \hat{F}_0 = \hat{F}_{0z} + \hat{F}_{0x}, \hat{F}_l = \hat{F}_{lz} + \hat{F}_{lx}$$

From expressions (1.10) and (1.1) we obtain

$$\hat{v} = \sum 2\pi n v l^{-2} \sin(\lambda_n y)\hat{I}_n(t)$$

$$\hat{I}_n(t) = \int_0^t \hat{u}(t-\tau)\exp[(-\nu\lambda_n^2 - i2\Omega)\tau]d\tau \tag{1.12}$$

$$\hat{F}_0 = 2\pi^2\rho\nu^2 l^{-3}\sum n^2\hat{I}_n(t), \quad \hat{F}_l = 2\pi^2\rho\nu^2 l^{-3}\sum(-1)^n n^2\hat{I}_n(t) \tag{1.13}$$

Taking into account the well-known formula in [5, p. 378], we can express the sum of the series in terms of elliptic functions

$$\sum 2\pi n l^{-2}\sin(\lambda_n y)\exp(-\lambda_n^2 t) = \sum \Psi(2nl + y, t) = -\frac{1}{l}\frac{\partial}{\partial y}\theta_3\left(\frac{y}{2l}, \frac{t}{l^2}\right)$$

$$\Psi(y, t) = \frac{y}{2\sqrt{\pi t}}\exp\left(\frac{-y^2}{4t}\right), \quad t > 0; \quad \theta_3(v, t) = 1 + 2\sum\exp(-n^2\pi^2 t)\cos 2\pi n v$$

For example, the velocity field can be expressed in terms of elliptic Jacobi theta-functions

$$\hat{v} = -\frac{\nu}{l}\int_0^t \exp(-2i\Omega\tau)\hat{u}(t-\tau)\frac{\partial}{\partial y}\theta_3\left(\frac{y}{2l}, \frac{\tau\nu}{l^2}\right)d\tau$$

and can also be represented in the form of a series

$$\hat{v} = \nu\sum\int_0^t \Psi(2nl + y, \nu\tau)\hat{u}(t-\tau)\exp(-2i\Omega\tau)d\tau$$

Note that the result obtained in [4], corresponding to the case of a rotating half-space of a viscous fluid above a flat plate, is obtained from the last expression when $n = 0$.

2. THE VELOCITY FIELD OF A FLOW INDUCED BY A SPECIFIED MOTION OF THE BOUNDARY PLANE

Suppose one of the planes, forming the boundary of the gap, oscillates with attenuation in a longitudinal direction with velocity

$$\hat{u} = \exp(-\alpha t)\sum_{\pm} \hat{A}_{\pm}\exp(\pm i\omega t) \tag{2.1}$$

where $\hat{A}_{\pm}$ are complex amplitudes (constant quantities), ω is the frequency and α is the attenuation factor. In this case, the velocity field and the shear stresses can be represented in the form

$$\hat{v} = \frac{2\pi\nu}{l^2}\exp(-2i\Omega t)\sum n\sin(\lambda_n y)S_n(t) \tag{2.2}$$

$$\hat{F}_0 = \frac{2\pi^2\rho\nu^2}{l^3}\exp(-2i\Omega t)\sum n^2 S_n(t), \quad \hat{F}_l = \frac{2\pi^2\rho\nu^2}{l^3}\exp(-2i\Omega t)\sum(-1)^n n^2 S_n(t) \tag{2.3}$$

Here

$$S_n(t) = \sum_{\pm}\hat{A}_{\pm}\frac{\exp(p_{\pm}t) - \exp(-\nu\lambda_n^2 t)}{\nu\lambda_n^2 + p_{\pm}} = 2\sum_{\pm}\hat{A}_{\pm}\frac{\exp[-(\nu\lambda_n^2 - p_{\pm})t/2]}{\nu\lambda_n^2 + p_{\pm}}\operatorname{sh}\frac{\nu\lambda_n^2 + p_{\pm}}{2}$$

$$p_{\pm} = -\alpha + i(2\Omega \pm \omega)$$

It can be seen that, even in the case of resonance frequencies ω, i.e. when $\nu\lambda_n^2 + p_\pm = 0$, the velocity field nevertheless remains continuous.

It is of interest to consider the behaviour of the solution as $t \to \infty$. Taking into account the fact that $\lambda_n^2 \approx n^2$, we will use tabulated sums given in [6]. Then, we obtain the following asymptotic expressions for the velocity field and the shear stresses

$$\hat{v} = \exp(-\alpha t)\sum_{\pm}\left[\hat{A}_\pm \exp(\pm i\omega t)\frac{\mathrm{sh}(l-y)\sigma_\pm}{\mathrm{sh}(l\sigma_\pm)}\right] \tag{2.4}$$

$$\hat{F}_0 = -\rho\nu\exp(-\alpha t)\sum_{\pm}\hat{A}_\pm\sigma_\pm\exp(\pm i\omega t)\mathrm{cth}(l\sigma_\pm) \tag{2.5}$$

$$\hat{F}_l = -\rho\nu\exp(-\alpha t)\sum_{\pm}\hat{A}_\pm\sigma_\pm\frac{\exp(\pm i\omega t)}{\mathrm{sh}(l\sigma_\pm)} \tag{2.6}$$

where $\sigma_\pm = \sqrt{p_\pm/\nu}$. Taking the limit in relations (2.4) and (2.6) as $l \to \infty$ (the gap is then converted into a half-space), we obtain exactly formulae (2.17) and (2.18) from [4], where the motion of a viscous fluid in a half-space bounded by a plane moving with velocity (2.1) was considered.

We will now show that not only in the asymptotic case but also in the general case, the velocity field (1.12) changes into the corresponding field for a half-space [4] as $l \to \infty$. In fact, in the case of a gap of width l we obtain from the boundary conditions a discrete set of eigenvalus $\lambda_n = \pi n/l$. As $l \to \infty$ this set constitutes a continuous spectrum $0 < \lambda < \infty$. Hence, the sum on the right-hand side of (1.2) is replaced by an integral as $l \to \infty$. Taking this observation into account, we can write the velocity field (1.12) in the form

$$\hat{v} = \frac{2\nu}{\pi}\int_0^t \hat{u}(t-\tau)\exp(-2i\Omega\tau)\left[\int_0^\infty \lambda\sin(\lambda y)\exp(-\nu\lambda_n^2 t)d\lambda\right]d\tau \tag{2.7}$$

Evaluating the non-singular integral, we obtain the formula for $\hat{v}$ from [4].

3. THE STRUCTURE OF THE BOUNDARY LAYERS

We will investigate expression (2.2) for the velocity field in more detail. It can be seen from formula (2.2) that, as one approaches the fixed boundary of the gap along the y axis, i.e. as $y \to l$, one observes an exponential attenuation of the fluid velocity at any instant of time.

Consider the fluid velocity field (2.4). We will represent the expressions for the frequencies $\sigma_\pm$ in (2.4) in the form

$$\sigma_\pm = \frac{1}{\delta_\pm} + \xi_\pm,\quad \delta_\pm = \left(\frac{2\nu}{|p_\pm| - \alpha}\right)^{1/2},\quad \xi_\pm = \left(\frac{|p_\pm| + \alpha}{2\nu}\right)^{1/2} \tag{3.1}$$

We can write the formula for the fluid velocity (2.4) as follows:

$$\hat{v} = \exp(-\alpha t)\sum_{\pm}\hat{A}_\pm\exp(\pm i\omega t)\,[\mathrm{ch}(\sigma_\pm y) - \mathrm{sh}(\sigma_\pm y)\mathrm{cth}(\sigma_\pm l)] \tag{3.2}$$

Representing the hyperbolic sine and hyperbolic cosine in terms of exponential functions, we see that the solution represents a set of eight waves with wave numbers $\xi_\pm$ and frequency ω, propagating along the y axis in the opposite direction to one another and decaying exponentially with time. When $l \gg \delta_\pm$ all the waves attenuate exponentially along the y axis at distances $\delta_\pm$ respectively. Hence, when $l \gg \delta_\pm$ the effect of the second plane, which bounds the gap, on the thicknesses of the boundary layers $\delta_\pm$ is not perceptible. Its presence in this case only leads to the occurrence of reflected waves, some of which have an amplitude $\hat{A}_\pm$ cth $l(1/\delta_\pm + i\xi_\pm)$, which depends on the width of the gap and the parameters of motion of the plane constituting its boundary.

Note that in the resonance case, i.e. when $\omega = 2\Omega$ we have $p_+ = -\alpha + 4i\Omega, p_- = -\alpha$, and therefore $1/\delta_- = 0$, $\xi_- = \sqrt{\alpha/\nu}$. Consequently, all the waves in solution (3.2) with amplitude $\hat{A}$ participate only in the oscillatory motion and do not decay along the y axis. If in this case $\alpha = 0$, we have $p_- = \xi_- = 0$, i.e. in the resonance case there are no waves with amplitude $\hat{A}_-$ travelling along the moving plane.

Consider the case of an exponentially decaying motion of the plane. We put $\omega = 0$ in (2.4)–(2.6). Then $p_+ = p_- = -\alpha + 2i\Omega = p$; we also put $\hat{A}_+ = \hat{A}_- = \hat{A}$. We obtain

$$\begin{gathered}\hat{v} = \hat{A}\exp(-\alpha t)\operatorname{sh}[(l-y)\sigma]/\operatorname{sh}(l\sigma)\\ \hat{F}_1 = -\rho\nu\sigma\hat{A}\exp(-\alpha t)/\operatorname{sh}(l\sigma), \quad \hat{F}_0 = -\rho\nu\sigma\hat{A}\exp(-\alpha t)/\operatorname{cth}(l\sigma)\end{gathered} \tag{3.3}$$

where $\sigma = \sqrt{p/\nu}$. From formulae (3.3) as $l \to \infty$, we obtain formulae (2.24) and (2.25) from [4].

Note that the difficulties involved in solving the problem when $\omega \to 2\Omega$ were pointed out in [2]. First, it turned out that one of the boundary layers, which is formed on the boundary of the gap, increases without limit as $\omega \to 2\Omega$. Second, the solution (the velocity field) depends on the order of the operations when one takes the double limit $t \to \infty$. $\omega \to 2\Omega$. When considering the Ekman boundary layers it was found [3] that the solution has a resonance behaviour as $\omega \to 2\Omega$; numerical results showed an increase in the flow velocity in the boundary layer due to this resonance.

Hence, our analysis has shown that attenuation ensures the existence of a solution inside the gap and reduces the difficulties pointed out previously in [2, 3].

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Translated by R.C.G.